Corrigé exercices "techniques de calcul"

Exercice I,1-1

a) $f(x) = \sqrt{8x^2 - 2x - 15} + \dfrac{4x}{\sqrt{10 - 3x}}$

conditions:

♦ $8x^2 - 2x - 15 \geq 0$

$\Delta = 4 + 480 = 484; x_1 = \dfrac{2+22}{16} = \dfrac{24}{16} = \dfrac{3}{2}; x_1 = \dfrac{2-22}{16} = -\dfrac{20}{16} = -\dfrac{5}{4}$

x	$-\infty$		$-\frac{5}{4}$		$\frac{3}{2}$		$+\infty$
$8x^2 - 2x - 15$		$+$	0	$-$	0	$+$	

donc $x \in \left]-\infty; -\frac{5}{4}\right] \cup \left[\frac{3}{2}; +\infty\right[$

♦ $10 - 3x > 0 \Leftrightarrow 3x < 10 \Leftrightarrow x < \dfrac{10}{3}$

finalement $\text{dom } f = \left]-\infty; -\frac{5}{4}\right] \cup \left[\frac{3}{2}; \frac{10}{3}\right[$

b) $g(x) = \dfrac{\cos 3x}{\tan^2 x - 3}$

♦ $x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ (domaine de la fonction tangente)

♦ $\tan^2 x - 3 \neq 0 \Leftrightarrow \tan^2 x \neq 3$

$\Leftrightarrow \tan x \neq \sqrt{3}$ et $\tan x \neq -\sqrt{3}$

$\Leftrightarrow x \neq \frac{\pi}{3} + k\pi, k \in \mathbb{Z}$ et $x \neq -\frac{\pi}{3} + k\pi, k \in \mathbb{Z}$

donc $\text{dom } g = \mathbb{R} \backslash \{\frac{\pi}{2} + k\pi; \frac{\pi}{3} + k\pi; -\frac{\pi}{3} + k\pi \mid k \in \mathbb{Z}\}$

Exercice I,2a-1

$f(x) = \dfrac{2x^2 + 5x - 12}{-x^2 - 3x + 4}$

a) condition: $-x^2 - 3x + 4 \neq 0 \overset{\Delta=25}{\Leftrightarrow} x \neq \frac{3+5}{-2} = -4$ et $x \neq \frac{3-5}{-2} = 1$ donc $\text{dom } f = \mathbb{R} \backslash \{-4; 1\}$

b) ♦ $\lim\limits_{x \to \pm\infty} \dfrac{2x^2 + 5x - 12}{-x^2 - 3x + 4} = \lim\limits_{x \to \pm\infty} \dfrac{2x^2}{-x^2} = -2$

♦ $\lim\limits_{x \to -4} \dfrac{\overbrace{2x^2 + 5x - 12}^{\to 0}}{\underbrace{-x^2 - 3x + 4}_{\to 0}}$ f.i. "$\dfrac{0}{0}$"

$\lim\limits_{x \to -4} \dfrac{2x^2 + 5x - 12}{-x^2 - 3x + 4} \overset{(H)}{=} \lim\limits_{x \to -4} \dfrac{4x + 5}{-2x - 3} = \dfrac{-11}{5} = -\dfrac{11}{5}$

♦ $\lim\limits_{x \to 1} \dfrac{\overbrace{2x^2 + 5x - 12}^{\to -5}}{\underbrace{-x^2 - 3x + 4}_{\to 0}}$ il faut calculer les limites à gauche et à droite

x	$-\infty$		-4		1		$+\infty$
$-x^2 - 3x + 4$		$-$	0	$+$	0	$-$	

donc $\lim\limits_{x \to 1^-} \dfrac{\overbrace{2x^2 + 5x - 12}^{\to -5}}{\underbrace{-x^2 - 3x + 4}_{\to 0^+}} = -\infty$ et $\lim\limits_{x \to 1^+} \dfrac{\overbrace{2x^2 + 5x - 12}^{\to -5}}{\underbrace{-x^2 - 3x + 4}_{\to 0^-}} = +\infty$ donc $\lim\limits_{x \to 1} \dfrac{2x^2 + 5x - 12}{-x^2 - 3x + 4}$ n'existe pas

Exercice I,2a-2

a) ♦ $\lim\limits_{x \to -2}(-x^3 + 2x^2 - 4x + 5) = -(-2)^3 + 2 \cdot (-2)^2 - 4 \cdot (-2) + 5 = 8 + 8 + 8 + 5 = 29$

♦ $\lim\limits_{x \to -\infty}(-x^3 + 2x^2 - 4x + 5) = \lim\limits_{x \to -\infty} -x^3 = +\infty$

d) $\lim\limits_{x \to 0} \dfrac{\overbrace{\cos x}^{\to 1}}{\underbrace{x}_{\to 0}}$ il faut calculer les limites à gauche et à droite

Exercice I,2b-1

$f(x) = \dfrac{3x^2 + 7x + 2}{-2x^2 - x + 6}$

a) condition: $-2x^2 - x + 6 \neq 0 \overset{\Delta=49}{\Longleftrightarrow} x \neq \frac{1+7}{-4} = -2$ et $x \neq \frac{1-7}{-2} = \frac{3}{2}$ donc $\text{dom } f = \mathbb{R} \backslash \{-2; \frac{3}{2}\}$

b) ♦ $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \dfrac{3x^2}{-2x^2} = -\dfrac{3}{2}$ $\boxed{\text{A.H.D. et A.H.G.}: y = -\frac{3}{2}}$

♦ $\lim\limits_{x \to -2} \dfrac{\overbrace{3x^2 + 7x + 2}^{\to 0}}{\underbrace{-2x^2 - x + 6}_{\to 0}}$ f.i. "$\frac{0}{0}$" il faut factoriser

$\lim\limits_{x \to -2} f(x) = \lim\limits_{x \to -2} \dfrac{6x + 7}{-4x - 1} = \dfrac{-5}{7} = -\dfrac{5}{7}$ $\boxed{\text{courbe privée du point } (-2; -\frac{5}{7})}$

♦ $\lim\limits_{x \to \frac{3}{2}} \dfrac{\overbrace{3x^2 + 7x + 2}^{\to \frac{77}{4}}}{\underbrace{-2x^2 - x + 6}_{\to 0}}$ il faut calculer les limites à gauche et à droite $3 \cdot \frac{9}{4} + 7 \cdot \frac{3}{2} + 2 = \frac{77}{4}$

x	$-\infty$		-2		$\frac{3}{2}$		$+\infty$
$-2x^2 - x + 6$		$-$	0	$+$	0	$-$	

$\lim\limits_{x \to \frac{3}{2}^-} \dfrac{\overbrace{3x^2 + 7x + 2}^{\to \frac{77}{4}}}{\underbrace{-2x^2 - x + 6}_{\to 0^+}} = +\infty$ et $\lim\limits_{x \to \frac{3}{2}^+} \dfrac{\overbrace{3x^2 + 7x + 2}^{\to \frac{77}{4}}}{\underbrace{-2x^2 - x + 6}_{\to 0^-}} = -\infty$ donc $\boxed{\text{A.V.}: x = \frac{3}{2}}$

Exercice I,3-4

a) $f(x) = 3x^4 - 2x^3 + \dfrac{5}{2}x^2 - x + 2\sqrt{3}$

$\text{dom } f = \text{dom } f' = \mathbb{R}$

$(\forall x \in \text{dom } f') : f'(x) = 3 \cdot 4x^3 - 2 \cdot 3x^2 + \dfrac{5}{2} \cdot 2x - 1 = 12x^3 - 6x^2 + 5x - 1$

b) $f(x) = -5(7x^2 + 4)^4 + 7$

$\text{dom } f = \text{dom } f' = \mathbb{R}$

$(\forall x \in \text{dom } f') : f'(x) = -5 \cdot 4(7x^2 + 4)^4 \cdot 7 \cdot 2x = -280x(7x^2 + 4)^3$

c) $f(x) = \dfrac{-3}{(5x+2)^2} = -3(5x+2)^{-2}$

condition: $5x+2 \neq 0 \Leftrightarrow x \neq -\dfrac{2}{5}$

$\text{dom } f = \text{dom } f' = \mathbb{R} \setminus \left\{-\dfrac{2}{5}\right\}$

$(\forall x \in \text{dom } f') : f'(x) = -3 \cdot (-2)(5x+2)^{-3} \cdot 5 = \dfrac{30}{(5x+2)^3}$

d) $f(x) = \dfrac{5+2x}{1-3x}$

condition: $1-3x \neq 0 \Leftrightarrow x \neq \dfrac{1}{3}$

$\text{dom } f = \text{dom } f' = \mathbb{R} \setminus \left\{\dfrac{1}{3}\right\}$

$(\forall x \in \text{dom } f') : f'(x) = \dfrac{2(1-3x)-(5+2x)(-3)}{(1-3x)^2} = \dfrac{2-6x+15+6x}{(1-3x)^2} = \dfrac{17}{(1-3x)^2}$

e) $f(x) = \overbrace{2x^2}^{u}\overbrace{\sqrt{x+1}}^{v}$

condition: $x+1 \geq 0 \Leftrightarrow x \geq -1$

$\text{dom } f = [-1;+\infty[\qquad \text{dom } f' =]-1;+\infty[$

$(\forall x \in \text{dom } f') : f'(x) = \overbrace{4x}^{u'}\overbrace{\sqrt{x+1}}^{v} + \overbrace{2x^2}^{u}\overbrace{\dfrac{1}{2\sqrt{x+1}}}^{v'} = 4x\sqrt{x+1} + \dfrac{x^2}{\sqrt{x+1}}$

$= \dfrac{4x(x+1)+x^2}{\sqrt{x+1}} = \dfrac{4x^2+4x+x^2}{\sqrt{x+1}} = \dfrac{5x^2+4x}{\sqrt{x+1}}$

f) $f(x) = \dfrac{\overbrace{x^2-x}^{u}}{\underbrace{\sqrt{x}}_{v}}$

condition: $x > 0$

$\text{dom } f = \text{dom } f' =]0;+\infty[$

$(\forall x \in \text{dom } f') : f'(x) = \dfrac{\overbrace{(2x-1)}^{u'}\overbrace{\sqrt{x}}^{v} - \overbrace{(x^2-x)}^{u}\overbrace{\frac{1}{2\sqrt{x}}}^{v'}}{\underbrace{x}_{v^2}} = \dfrac{\frac{2(2x-1)x-(x^2-x)}{2\sqrt{x}}}{x} = \dfrac{4x^2-2x-x^2+x}{2x\sqrt{x}} = \dfrac{3x^2-x}{2x\sqrt{x}} = \dfrac{3x-1}{2\sqrt{x}}$

g) $f(x) = x^2 \cdot \cos x - 2x \cdot \sin x$

$\text{dom } f = \text{dom } f' = \mathbb{R}$

$(\forall x \in \text{dom } f') : f'(x) = 2x\cos x + x^2(-\sin x) - 2\sin x - 2x\cos x = -x^2\sin x - 2\sin x = (-x^2-2)\sin x$

h) $f(x) = \overbrace{(3x+4)^3}^{u}\overbrace{(5-4x)^4}^{v}$

$\text{dom } f = \text{dom } f' = \mathbb{R}$

$(\forall x \in \text{dom } f') : f'(x) = \overbrace{3(3x+4)^2 \cdot 3}^{u'} \cdot \overbrace{(5-4x)^4}^{v} + \overbrace{(3x+4)^3}^{u} \cdot \overbrace{4(5-4x)^3 \cdot (-4)}^{v'}$

$= 9(3x+4)^2(5-4x)^4 - 16(3x+4)^3(5-4x)^3$

$= (3x+4)^2(5-4x)^3[9(5-4x) - 16(3x+4)]$

$= (3x+4)^2(5-4x)^3(45-36x-48x-64)$

$= (3x+4)^2(5-4x)^3(-84x-19)$

Exercice II,1-1

$f(x)=\dfrac{12x}{(x+2)^2}$

a) C.E.: $(x+2)^2\neq 0 \Leftrightarrow x\neq -2$ $\qquad$ $\text{dom } f=\text{dom } f'=\text{dom } f''=\mathbb{R}\backslash\{-2\}$

b) ♦ $\displaystyle\lim_{x\to-2}\frac{\overbrace{12x}^{\to -24}}{\underbrace{(x+2)^2}_{\to 0^+}}=-\infty$ $\qquad$ A.V.: $x=-2$

♦ $\displaystyle\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{12x}{x^2}=\lim_{x\to+\infty}\frac{12}{x}=0$ $\qquad$ A.H.D.: $y=0$

♦ $\displaystyle\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{12x}{x^2}=\lim_{x\to-\infty}\frac{12}{x}=0$ $\qquad$ A.H.G.: $y=0$

c) $(\forall x\in\text{dom } f')$: $f'(x)=\dfrac{12(x+2)^2-12x\cdot 2(x+2)}{(x+2)^4}=\dfrac{12(x+2)(x+2-x\cdot 2)}{(x+2)^4}=\dfrac{12(2-x)}{(x+2)^3}$

$f'(x)=0\Leftrightarrow 2-x=0\Leftrightarrow x=2$

x	$-\infty$		-2		2		$+\infty$
$12(2-x)$		$+$		$+$	0	$-$	
$(x+2)^3$		$-$	0	$+$		$+$	
$f'(x)$		$-$	$\Vert$	$+$	0	$-$	

$f(2)=\dfrac{12\cdot 2}{(x+2)^2}=\dfrac{24}{16}=\dfrac{3}{2}$ $\qquad$ maximum $(2;\frac{3}{2})$

d) $(\forall x\in\text{dom} f'')$: $f''(x)=\dfrac{-12(x+2)^3-12(2-x)\cdot 3\cdot(x+2)^2}{(x+2)^6}=\dfrac{-12(x+2)^2(x+2+6-3x)}{(x+2)^6}$

$=\dfrac{-12(-2x+8)}{(x+2)^4}=\dfrac{24(x-4)}{(x+2)^4}$

Le signe de f” est celui de $x-4$.

x	$-\infty$		-2		4		$+\infty$
$f''(x)$		$-$	$\Vert$	$-$	0	$+$	

$f(4)=\dfrac{12\cdot 4}{(x+4)^2}=\dfrac{48}{36}=\dfrac{4}{3}$ $\qquad$ point d’inflexion $(4;\frac{4}{3})$

e) Tableau de variations et de concavité:

x	$-\infty$		-2		2		4		$+\infty$
f'		$-$	$\Vert$	$+$		$-$		$-$	
f''		$-$	$\Vert$	$-$	0	$-$		$+$	
f	0	$\searrow$	$-\infty\ \Vert\ -\infty$	$\nearrow$	$\frac{3}{2}$	$\searrow$		$\searrow$	0
$\mathcal{C}_f$		$\frown$	$\Vert$	$\frown$		$\frown$	p.i.	$\smile$	

f) www.wolframalpha.com

g) (t_1) : $y=f'(1)(x-1)+f(1)\Leftrightarrow y=\dfrac{12}{27}(x-1)+\dfrac{12}{9}\Leftrightarrow y=\dfrac{4}{39}x+\dfrac{8}{9}$

Exercice II,1-2

a) $\displaystyle\lim_{x\to0}\frac{\overbrace{x+\sin(2x)}^{\to 0}}{\underbrace{x-\sin(2x)}_{\to 0}}\overset{(H)}{=}\lim_{x\to0}\frac{1+2\cos(2x)}{1-2\cos(2x)}=-3$

b) $\displaystyle\lim_{x\to2}\frac{\overbrace{x^3-7x^2+16x-12}^{\to 0}}{\underbrace{x^3-6x^2+12x-8}_{\to 0}}\overset{(H)}{=}\lim_{x\to2}\frac{\overbrace{3x^2-14x+16}^{\to 0}}{\underbrace{3x^2-12x+12}_{\to 0}}\overset{(H)}{=}\lim_{x\to2}\frac{\overbrace{6x-14}^{\to -2}}{\underbrace{6x-12}_{\to 0}}$ $\qquad$ il faut distinguer 2^+ et 2^-

$\displaystyle\lim_{x\to2^+}\frac{\overbrace{6x-14}^{\to -2}}{\underbrace{6x-12}_{\to 0^+}}=-\infty$ $\qquad$ et $\qquad$ $\displaystyle\lim_{x\to2^-}\frac{\overbrace{6x-14}^{\to -2}}{\underbrace{6x-12}_{\to 0^-}}=+\infty$ $\qquad$ donc cette limite n’existe pas.

Exercice II,1-3

a) C.E.: $x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ dom f = dom f′ $= \mathbb{R}\setminus\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

$(\forall x \in \text{dom} f')\ :\ f'(x) = 2x + 2\tan x(1+\tan^2 x) - 2\tan x - 2x(1+\tan^2 x)$

$= 2x + 2\tan x + 2\tan^3 x - 2\tan x - 2x - 2x\tan^2 x$

$= 2\tan^3 x - 2x\tan^2 x = 2\tan^2 x(\tan x - x)$

b) C.E.: $2x > 0 \Leftrightarrow x > 0$ dom f = dom f′ $=]0;+\infty[$

$$(\forall x \in \text{dom} f')\ :\ f'(x) = \frac{(2x-2)\sqrt{2x} - (x^2-2x)\cdot\dfrac{2}{2\sqrt{2x}}}{2x} = \frac{(2x-2)\sqrt{2x} - \dfrac{(x^2-2x)}{\sqrt{2x}}}{2x}$$

$$= \frac{(2x-2)(2x)-(x^2-2x)}{2x\sqrt{2x}} = \frac{4x^2-4x-x^2+2x}{2x\sqrt{2x}} = \frac{3x^2-2x}{\sqrt{2x}\,2x} = \frac{3x-2}{2\sqrt{2x}}$$

Exercice II,2-1

$$f(x) = \frac{x^2-1}{(x-2)^2}$$

a) C.E.: $(x-2)^2 \neq 0 \Leftrightarrow x \neq 2$ dom f = dom f′ = dom f″ $= \mathbb{R}\setminus\{2\}$

b) ♦ $\lim\limits_{x\to 2} \dfrac{\overbrace{x^2-1}^{\to 3}}{\underbrace{(x-2)^2}_{\to 0^+}} = +\infty$ A.V.: $x = 2$

♦ $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \dfrac{x^2}{x^2} = 1$ A.H.D.: $y = 1$ ♦ $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x^2}{x^2} = 1$ A.H.G.: $y = 1$

c) ♦ $(\forall x \in \text{dom} f')\ :\ f'(x) = \dfrac{2x(x-2)^2 - (x^2-1)\cdot 2(x-2)}{(x-2)^4} = \dfrac{(x-2)[2x(x-2)-2(x^2-1)]}{(x-2)^4} = \dfrac{-4x+2}{(x-2)^3}$

$f'(x) = 0 \Leftrightarrow -4x+2 = 0 \Leftrightarrow x = \frac{1}{2}$

x	$-\infty$		$\frac{1}{2}$		2		$+\infty$
$-4x+2$		+	0	−		−	
$(x-2)^3$		−		−	0	+	
$f'(x)$		−	0	+	‖	−	

$f\left(\frac{1}{2}\right) = \dfrac{\left(\frac{1}{2}\right)^2-1}{\left(\frac{1}{2}-2\right)^2} = -\dfrac{1}{3}$ minimum $E(\frac{1}{2};-\frac{1}{3})$

♦

$$(\forall x \in \text{dom} f'')\ :\ f''(x) = \frac{-4(x-2)^3 - (-4x+2)\cdot 3\cdot (x-2)^2}{(x-2)^6} = \frac{(x+2)^2[-4(x-2)-3(-4x+2)]}{(x-2)^6} = \frac{8x+2}{(x-2)^4}$$

$f''(x) = 0 \Leftrightarrow 8x+2 = 0 \Leftrightarrow x = -\dfrac{1}{4}$

Le signe de f” est celui de $8x+2$.

x	$-\infty$		$-\frac{1}{4}$		2		$+\infty$
$f''(x)$		−	0	+	‖	+	

$f\left(-\frac{1}{4}\right) = \dfrac{\left(-\frac{1}{4}\right)^2-1}{\left(-\frac{1}{4}-2\right)^2} = -\dfrac{5}{27}$ point d’inflexion

$I(-\frac{1}{4};-\frac{5}{27})$

Exercice II,2-2

$f(x) = \sqrt{\dfrac{-2x-3}{5+4x}}$ C. E. : $\dfrac{-2x-3}{5+4x} \geq 0$ $-2x-3 = 0 \Leftrightarrow x = -\dfrac{3}{2}$ et $4x+5 = 0 \Leftrightarrow x =$

x	$-\infty$		$-\frac{3}{2}$		$-\frac{5}{4}$		$+\infty$
$-2x-3$		$+$	0	$-$		$-$	
$5+4x$		$-$		$-$	0	$+$	
$f'(x)$		$-$	0	$+$	$\|$	$-$	

donc $\operatorname{dom}f = [-\dfrac{3}{2}; -\dfrac{5}{4}[$ et $\operatorname{dom}f' =]-\dfrac{3}{2}; -\dfrac{5}{4}[$

$$(\forall x \in \operatorname{dom}f') : f'(x) = \frac{\dfrac{-2(5+4x)-(-2x-3)\cdot 4}{(5+4x)^2}}{2\sqrt{\dfrac{-2x-3}{5+4x}}} = \sqrt{\frac{5+4x}{-2x-3}} \cdot \frac{-10-8x+8x+12}{2(5+4x)^2} = \sqrt{\frac{5+4x}{-2x-3}} \cdot \frac{1}{(5+4x)^2}$$

$f' > 0$, donc f est strictement croissante sur son domaine.

Exercice II,2-3

a) $\displaystyle \lim_{x\to 0} \frac{\overbrace{\sin^2(3x)}^{\to 0}}{\underbrace{3x^2}} \overset{(H)}{=} \lim_{x\to 0} \frac{2\sin(3x)\cdot\cos(3x)\cdot 3}{3\cdot 2x^2} = \lim_{x\to 0} \frac{\overbrace{\sin(3x)}^{\to 0}\cdot\cos(3x)}{\underbrace{x}_{\to 0}}$

$\displaystyle \overset{(H)}{=} \lim_{x\to 0} \frac{\overset{\to 0}{\cos(3x)\cdot 3\cdot\cos(3x)+\sin(3x)\cdot(-\sin(3x))\cdot 3}}{1} = \lim_{x\to 0} 3(\underbrace{\cos^2(3x)}_{\to 1} - \underbrace{\sin^2(3x)}_{\to 0}) = 3$

b) $\displaystyle \lim_{x\to 0} \frac{\overbrace{x^2-1}^{\to -1}}{\underbrace{\tan x}}$ il faut distinguer 0^+ et 0^-

$\displaystyle \lim_{x\to 0^-} \frac{\overbrace{x^2-1}^{\to -1}}{\underbrace{\tan x}_{\to 0^-}} = +\infty$ et $\displaystyle \lim_{x\to 0^+} \frac{\overbrace{x^2-1}^{\to -1}}{\underbrace{\tan x}_{\to 0^+}} = -\infty$ donc cette limite n'existe pas.

Exercice II,2-4

C.E.: $1-\sin x \neq 0 \Leftrightarrow \sin x \neq 1 \Leftrightarrow x \neq \frac{\pi}{2} + 2k\pi$ $\operatorname{dom}f = \operatorname{dom}f' = \mathbb{R}\backslash\{\frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\}$

$\displaystyle (\forall x \in \operatorname{dom}f') : f'(x) = \frac{3(-\sin x)\cdot(1-\sin x) - 3\cos x(-\cos x)}{(1-\sin x)^2} = \frac{-3\sin x + 3\sin^2 x + 3\cos^2 x}{(1-\sin x)^2}$

$\displaystyle = \frac{3-3\sin x}{(1-\sin x)^2} = \frac{3(1-\sin x)}{(1-\sin x)^2} = \frac{3}{1-\sin x}$